A vector has both a magnitude and a direction. A scalar has only a magnitude, no direction. Vectors are indicated by an arrow over the symbol, e.g. the velocity vector is written as $\vec{v}$. Vectors are represented by arrows. Length of arrow = magnitude of vector.

Examples:

Vectors: acceleration, velocity, displacement, force, electric field.

Scalars: speed, distance traveled, time, electric potential

Differential Geometry

Position vector: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ or $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Velocity: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

Arc length: $s(t) = \int_{t_1}^{t_2} |\frac{d\vec{r}}{dt}|\, dt, \quad \frac{ds}{dt} = |\vec{R}'(t)| = |\vec{v}(t)|$

Acceleration: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$

DIFFERENTIATION OF VECTOR VALUED FUNCTIONS:

Q 1. If $\vec{r} = (t^3 + t^2 + t)i + (t^2 + t)j + (t+1)k$ then find $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.

Q 2. If $\vec{r} = a\cos ti + a\sin tj + tk$ then find $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$ and. $\left|\frac{d^2\vec{r}}{dt^2}\right|$.

Q 3. If $\vec{u} = t^2 i - tj + (2t+1)k$ and $\vec{v} = (2t-3)i + j - tk$ then find $\frac{d}{dt}(u.v)$ at t=1 **[June 15]**

Q 4. A particle moves along the curve $x = t^3 + 1, y = t^2, z = 2t + 5$ then find velocity and acceleration at the time t=0 and t=1/2π. ***[Ans: v= 4j+6k , a=-4i and v=-4i+6k , a= -4j]***

Q 5. A particle moves along the curve $x = 4\cos t, y = 4\sin t, z = 6t$ then find the component of velocity and acceleration at the time t=1 in the direction of $i+j+k$.

[***Ans: v=3i+2j+2k , unit vector*** $\hat{n} = \frac{i+j+3k}{|i+j+3k|} = \frac{i+j+3k}{\sqrt{11}}$, Component of velocity=

$\vec{v}.\hat{n} = (3i+2j+3k).\frac{i+j+3k}{\sqrt{11}} = \sqrt{11}$, component of acceleration $= \vec{a}.\hat{n} = (6i+2j).\frac{i+j+3k}{\sqrt{11}} = \frac{8}{\sqrt{11}}$

Q 6. If $\vec{r}.d\vec{r} = 0$ be the unit vector in the direction of $\vec{r}$ show that $\hat{a} = \frac{\vec{r}}{|\vec{r}|} = \frac{xi + yj + zk}{r}$

Q 7. If $\vec{r} = xi + yj + zk = xi + axj + bxk$, and $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = const.$ then show that $\vec{r}$ =constant. [**Jan.2007**]

[Hint: $\vec{r} \times d\vec{r} = (ydz - zdy)i + (zdx - xdz)j + (xdy - ydz)k = 0 = 0i + 0j + 0k$, on solving we get $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$, after integration we get $y=ax$, $z=bx$, then $\vec{r} = xi + yj + zk = xi + axj + bxk$, and $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = const.$]

Integration of vector valued functions:

Q 8. If $\vec{r}(t) = 5t^2 i + tj - t^3 k$ then find $\int_1^2 \vec{r}dt$ ***[Ans:-5/6 i+14/3j-3k]***

Q 9. If $\vec{r}(t) = 5t^2 i + tj - t^3 k$ then prove that $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\, dt = -14i + 75j - 15k$ **[June2006]**

Q 10. **If** $\vec{r}(t) = \begin{cases} 2i - j + 2k\,, & t = 2 \\ 4i - 2j + 3k & ,t = 3 \end{cases}$ then show that $\int_2^3 \vec{r}.\frac{d\vec{r}}{dt}dt = 10$ [**Dec.2006**]

Q 11. Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = a$, given that at t=0 , $\vec{r} = 0$ and $\frac{d\vec{r}}{dt} = 0$

.[Ans: $\vec{r} = \frac{1}{2}at^2$

SCALAR AND VECTOR FIELDS:

Definition : A **scalar field** is a scalar valued function $\varphi(x,\ y,\ z)$ of three variables x, y, and z.
An example is

$$\varphi(x,\ y,\ z) = x^2 + y^2 + z^2 .$$

For this φ, if $\varphi(x,\ y,\ z) = c$, a constant, then it represents a sphere with centre at the origin.
Definition A **vector field** is a vector valued function $\mathbf{F}(x,\ y,\ z)$ of three variables. An example is

$$\mathbf{F}(x,\ y,\ z) = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2 + z^2}, \qquad \text{provided } x^2 + y^2 + z^2 \neq 0.$$

GRADIENT:

We note that the vector differential operator **del,** written ∇ is defined by

$$\boxed{\vec{\nabla}\ or\ \nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}}$$

The operator ∇ possesses properties like ordinary vectors.
Definition Let $\varphi(x,\ y,\ z)$ be a given differentiable scalar field, then the gradient of φ, written $\nabla\varphi$ is defined by

$$\boxed{\nabla\varphi \equiv \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\varphi \quad = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k}.}$$

We note that gradφ = $\nabla\varphi$ defines a vector field.

DIVERGENCE AND CURL

We have seen that gradient describes the rate of change of a scalar field. Now we consider the problem of describing the rate of change of a vector field. There are two fundamental measures of this rate of change: one is *divergence* and the other is *curl*. The divergence of a vector field is a scalar field and the curl of a vector field is a vector field.

DEFINITION: The **divergence** of a differentiable vector field $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ is a scalar field, denoted $\operatorname{div}\mathbf{F}$ or $\nabla\cdot\mathbf{F}$ defined by

$$\boxed{\operatorname{div}\mathbf{F} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} = \mathbf{i}\cdot\frac{\partial \mathbf{F}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{F}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{F}}{\partial z} = \nabla\cdot\mathbf{F}}.$$

$\nabla\cdot\mathbf{F}$ is read del dot **F**. We note that $\nabla\cdot\mathbf{F}$ is a scalar valued function.

- *A vector is **solenoidal** if its divergence is zero.*

DEFINITION: *The* **curl** of a vector field $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ is the vector field, denoted by curl **F** or $\nabla\times\mathbf{F}$, defined by

$$\boxed{\operatorname{curl}\vec{\mathbf{F}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \mathbf{i}\times\frac{\partial \vec{\mathbf{F}}}{\partial x}+\mathbf{j}\times\frac{\partial \vec{\mathbf{F}}}{\partial y}+\mathbf{k}\times\frac{\partial \vec{\mathbf{F}}}{\partial z}}.$$

$\nabla\times\mathbf{F}$ is read del cross **F**. We note that $\nabla\times\mathbf{F}$ is a vector valued function.

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- *A vector $\vec{\mathbf{F}}$ is **Irrotational** if its curl $\vec{\nabla}\times\vec{\mathbf{F}}$ is the zero vectors (i.e. $\vec{\nabla}\times\vec{\mathbf{F}} = 0$).*

- *A vector valued function* **F** *is a conservative field if it is **Irrotational**. (i.e. if curl **F = 0**).*

Vocabulary (arising from considering the vector field as fluid flow lines):

- When the curl is non-zero, the fluid tends to rotate about an axis parallel to the direction of the curl.
- When the curl is zero, the fluid does not tend to rotate, and thus is called "***irrotational***."
- When the divergence is positive, there is a "***source***"; more fluid flows into a region than flows out.
- When the divergence is negative, there is a "***sink***"; more fluid leaves a region than flows into it.

When the divergence is zero, we call the flow "***source-free***" or "***incompressible***" because the amount of fluid that flows into a region is equal to amount that flows out

Q.2. Find gradient of the scalar function $\phi(x,y,z) = x^2 + y^2 - z$ at the point (1,2,5). **[June16]**

Q.3. Define divergence of a vector point function and explain its meaning. **[June 16]**

Q.4. Show that (i) $div\,\vec{r} = 3$ (ii) $curl\,\vec{r} = 0$ (iii) $div\,\hat{r} = \frac{2}{r}$ (iv) $curl\,\hat{r} = 0$

Q.5. Show that (i) $grad\, r = \hat{r}$ (ii) $grad \frac{1}{r} = -\frac{\hat{r}}{r^2}$ (iii) $grad\, r^n = nr^{n-2}\vec{r}$ [Dec.11, **June15]**

Q.6. Prove $\nabla^2(r^n\vec{r}) = (n+3)r^{n-2}\vec{r}$

Q.7. Find $div(curl\,\vec{F})$,where $\vec{F} = x^2yi + xzj + 2yzk$. **[Nov. 2019]**

Q.8. Find $div\,\vec{F}$ and $curl\,\vec{F}$,when $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

Q.9. If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ prove that $\vec{F}.curl\,\vec{F} = 0$ **[June .2004,Jan.2006]**

Q.10. Prove that $div\, grad\, r^m = m(m+1)r^{m-2}$ **[June 2003,08, Dec. 2008 ,June15]**

Q.11. Find A if $div\, grad\, r^m = Ar^{m-2}$ **[June2004]**

Q.12. Prove (i) $\nabla f(r) = f'(r)\frac{\vec{r}}{r}$ (ii) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$ **[june 2001,Dec.2002,june2008]**

SOLENOIDAL AND IRROTATIONAL VECTORS :

Q.13. Show that $\vec{F} = (x+3y)i + (y-2z)j + (x-2z)k$ is solenoidal. **[Dec.2003,Dec.2010]**

Q.14. Show that the vector field $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ is Irrotational. **[June2007]**

Q.15. Prove that (1) $div\, r^n\vec{r} = (n+3)r^n$ (2) $curl\, r^n\vec{r} = \vec{0}$ (3) $r^n\vec{r}$ is solenoidal then n= -3.

Q.16. Show that the vector field $x^2 + y^2 = a^2$, $z = 0$ is Irrotational. **[June16]**

Q.17. Prove that $f(r)\vec{r}$ is Irrotational. [**Dec**.15]

Q.18. Show that the vector $\vec{F} = \frac{\vec{r}}{r^3}$ is Irrotational. Find the scalar potential. **[Dec.16]**

Q.19. Determine the constants *a*,*b*,*c*, so that $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is Irrotational. Find the scalar potential Φ such that $\vec{F} = grad\,\phi$ [Ans:*a*=*4*,*b*=*2*,*c*=*-1*]

UNIT NORMAL VECTOR:

(1) To a line passing through a point $\mathbf{\hat{a}} = \frac{\vec{a}}{|\vec{a}|} = \frac{xi + yj + zk}{\sqrt{x^2+y^2+z^2}}$

(2) To the surface φ is $\mathbf{\hat{a}} = \frac{grad\,\varphi}{|grad\,\varphi|} = \frac{\nabla.\varphi}{|\nabla.\varphi|}$ (if point is given then find $grad\,\varphi$ at the given point)

(3) To a line making an angle θ from x-axis $\mathbf{\hat{a}} = \frac{\cos\theta i + \sin\theta j}{|\cos\theta i + \sin\theta j|}$

(4) To a line segment $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$, $\hat{a} = \frac{li + mj + nk}{|\sqrt{l^2+m^2+n^2}|}$

Hence **unit normal vector at a point *P* in the direction of *φ*** is given by $\hat{n} = \dfrac{grad\,\varphi}{|grad\,\varphi|} = \dfrac{\nabla\varphi}{|\nabla\varphi|}$.

Q.2. Find the unit normal vector to the surface $x^4 - 3xyz + z^2 + 1$ at the point (1,1,1)

Q.3. Find the unit normal to the surface $xy^3z^2 = 4$ at the point (-1,-1,2) **[June07, feb.11]**

[Ans: $\hat{n} = \dfrac{grad\,\phi}{|grad\,\phi|} - \dfrac{i+3j-k}{\sqrt{11}}$

Q.4. The temperature of points in space is given by $T = x^2 + y^2 - z$. A mosquito located at (1,1,2) desire to fly in such direction that it will fly in such direction that it will fly in such direction that it will get warm as soon as possible .In what direction should it move.

[Hint: $\hat{n} = \dfrac{grad\,T}{|grad\,T|} = \dfrac{2i+2j-k}{3}$

DIRECTIONAL DERIVATIVE:

The **component of** $\nabla\varphi$ in the direction of a unit vector **a** is given by $\iiint_D f(x,y,z)dv = \iiint_D f(x,y,z)dxdydz$, and is called the **directional derivative** of φ in the direction of **a**. Physically this is the rate of change of φ at the point (*x*, *y*, *z*) in the direction of **a or** $\vec{a}$.

Another Definition for Directional Derivative

Let $\varphi(x, y, z)$ defines a scalar field in a region *R*. Let *AB* be a line segment passing through (x_0, y_0, z_0) parallel to a given unit vector **a**. Let *s* denote the displacement measured along the line segment in the direction of **a**, with *s* = 0 corresponding to (x_0, y_0, z_0). To each value of the parameter *s* there corresponds a point (*x*, *y*, *z*) on the line segment, and hence a corresponding scalar $\varphi(x, y, z)$. The derivative $\dfrac{d\varphi}{ds}$ at *s* = 0, if it exists, is called the *directional derivative* of φ at the point (x_0, y_0, z_0) in the direction of the vector **a**. Briefly speaking, the directional derivative of φ is the rate of change of φ per unit distance in some prescribed direction.

By the Chain Rule, since *x*, *y*, *z* are functions of *s*, we have

$$\frac{d\varphi}{ds} = \frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial z}\frac{dz}{ds} = \left(\frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k}\right).\left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) = \nabla\varphi.\frac{d\mathbf{r}}{ds}$$

Since $\dfrac{d\mathbf{r}}{ds}$ is a unit vector, and if we denote it by $\hat{\mathbf{a}}$, then

Directional derivative of a function φ in the direction of vector $\vec{a}$ is given by

$$D.D. = \frac{d\varphi}{ds} = grad\,\varphi.\hat{\mathbf{a}} = \nabla\varphi\cdot\hat{\mathbf{a}},$$

Remarks

1. If θ is the angle between gradφ and $\hat{\mathbf{a}}$, then

Directional derivative of a function φ = $\frac{d\varphi}{ds} = \nabla\varphi \cdot \mathbf{a} = |\nabla\varphi||\mathbf{a}|\cos\theta = |\nabla\varphi|\cos\theta$. (since $|\hat{\mathbf{a}}| = 1$)

It also follows that the **maximum value** of $\frac{d\varphi}{ds}$ is obtained when θ = 0 i.e. when gradφ and **a** are in the same direction and the maximum value is $|\nabla\varphi|$.

i.e ***the maximum value of directional derivative is in the direction of grad φ.***

2. If $\varphi(x, y, z) = c$ be a level surface through the point P (x_0, y_0, z_0). Then

$$\frac{d\varphi}{ds} = 0, \quad \text{so that} \quad \nabla\varphi.\frac{d\mathbf{r}}{ds} = 0. \qquad \ldots(1)$$

Since $\frac{d\mathbf{r}}{ds}$ is the unit tangent vector at (x_0, y_0, z_0), Eqn. (1) shows that $\nabla\varphi$ acts in a direction perpendicular to the direction of the tangent i.e., $\nabla\varphi$ acts along the normal to the surface at P.

Q.5. Find the directional derivative of $\phi = xy + yz + zx$ in the direction of the vector $i + 2j + 2k$ at the point (1,2,0). **[June15, Dec.16]**

Q.6. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point (2,-1,1) in the direction of the normal to the surface $x\log z - y^2 = -4$ at (-1,2,1).

Q.7. Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the pointP(1,1,1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. **[June 2008,Dec.15]**

[Ans: $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{li + mj + nk}{|li + mj + nk|} = \frac{2i - 2j + k}{3}$, $D.D. = \frac{35}{3}$

Q.8. Find the magnitude of directional derivative for the function $\phi = \frac{y}{x^2 + y^2}$ which makes an angle of 30^0 from the positive direction of x-axis at the point (0,1). **[Ans:-1/2]**

Q.9. In what direction from the point (2,1,-1) is the directional derivative of $\phi = x^2yz^3$ maximum and what is its magnitude?

[**Ans**: Required direction is $grad\phi = -4i - 4j + 12k$,maximum rate of change = $|grad\phi| = 4\sqrt{11}$

Q.10. Find the values of the constants a,b,c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1,2,-1) has a maximum magnitude 64 in the direction parallel to z-axis.

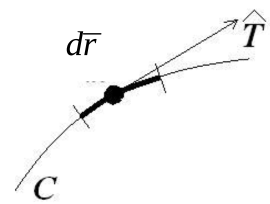
Q.11. Find the directional derivative of 1/r in the direction of $\vec{r}$ .. [Hint: $grad(\frac{1}{r}) = -\frac{\vec{r}}{r^3}$,

$D.D. = -\frac{1}{r^2}$

Q.12. Find the directional derivative of $1/r^2$. in the direction of $\vec{r}$. [Ans:, $D.D.=-\frac{2}{r^3}$]

VECTOR INTEGRATION:

LINE INTEGRAL: An integral which is evaluated along a curve .



$\exists$ **Curve C:** $\vec{R}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k},\ \ a\le t\le b$

$\exists$ Vector $\vec{F}(x,y,z)=F_1(x,y,z)\hat{i}+F_2(x,y,z)\hat{j}+F_3(x,y,z)\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r}=\int_a^b \vec{F}[x(t),y(t),z(t)]\cdot\vec{r}\,'(t)dt=\int_a^b \vec{F}.d\vec{r}\,,$$

Circulation: The integral of a vector $\vec{F}$ along a closed curve is called circulation of $\vec{F}$ round the closed curve C. i.e . $\int_C \vec{F}.dr$

Irrotational vector: A vector $\vec{F}$ is called Irrotational in the region **R** if the circulation of $\vec{F}$ round every closed curve in is zero. i.e. $\int_C \vec{F}.dr=0$.

Remark: When the path of integration is closed curve then notation $\oint$ is used in place of $\int$.

Q.13. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}=(x^2y^2)\hat{i}+y\hat{j}$ and the curve C is $y^2=4x$ in XY-plane from (0,0) to (4,4). {**Ans**: 264] **[June16]**

Q.14. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ and the curve C is arc of the helix $r=a\cos ti+b\sin tj+ctk$ from t=0 to π/2. [Hint : put $x=accost$,$y=bsint$, $z=ct$] **Ans:0**

Q.15. If $\vec{F}=2y\hat{i}-z\hat{j}+x\hat{k}$, evaluate $\int_C \vec{F}\times d\vec{R}$ along the curve $x=\cos t, y=\sin t, z=2\cos t$ from t=0 to π/2. [**Ans**: $\int_C \vec{F}\times d\vec{R}=(2-\frac{\pi}{4})i+(\pi-\frac{1}{2})j$] **[June 06, sept.09]**

Q.16. Find the total **work done** in moving a particle in a force field given by $\vec{F}=(x^2-y^2+x)\hat{i}-(2xy+y)\hat{j}$ in XY- Plane from (0,0) to (1,1) along the parabola y^2=x. **[Ans: $\int_c \vec{F}.d\vec{r}=-2/3$]**

Q.17. Find the total **work done** in moving a particle in a force field given by $\vec{F} = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$ along the curve $x = t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$. **[Ans:** $\int_c \vec{F}.d\vec{r} = 303$**] [Dec.2002]**

Q.18. IF $\vec{F} = 3xy\,\hat{i} - y\,\hat{j}$, evaluate $\int_C \vec{F}.d\,\vec{r}$,where C is the arc of the parabola $y=2x^2$,from (0,0) to (1,2). **[May. 2019]**

Q.19. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = (x^2+y^2)\,\hat{i} - 2xy\,\hat{j}$ and the curve C is a rectangle in XY-plane bounded by $x=0,x=a,y=0$, $y=b$.[*Hint:* $\int_C F.dr = 1/3a^3 - ab^2 - 1/3a^3 - b^2a = -2ab^2$]

Q.20. Find the circulation of F around the curve C where $\vec{F} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ and C is the rectangle whose vertices are (0,0),(1, π/2),(1,0),(0, π/2). ***[June 01,2003]***

.[***Hint***: $\int_C F.dr$

Q.21. Find $\int_C F.dr$, where $\vec{F} = (x^3 - yz)\,\hat{i} - zx^2y\,\hat{j} + z\,\hat{k}$,where C denotes the Curve bounded by the planes $x=0$,$x=a$,$y=0$, $y=a$, $z=0$, $z=a$

[**Ans:** $\int_C F.dr = (a^5 - \frac{1}{4}a^4) + (\frac{1}{4}a^4) - (\frac{2}{3}a^5) + (0) + (2a^2) - (2a^2) = \frac{1}{3}a^5$

Q.22. Find $\int_C F.dr$, where $\vec{F} = (x^2+y^2)\,\hat{i} - 2xy\,\hat{j}$ taken round the rectangle bounded by the line $x = \pm a$,$y=0$ $y=b$. [**Ans**: $\int_C \vec{F}.d\vec{r} = 0 + \frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{3}a^3 = \frac{1}{2}a^3$

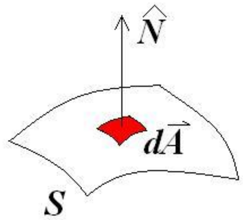
[Hint: Equation of line (two point Formula**:** $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = k(let)$

Q.23. Find $\int_C F.dr$, where $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x+z)k$ where C is the boundary of the triangle with vertices (0,0,0),(1,0 ,0) and (1,1,0). **[Hint:** Equation of line (two point Formula**:** $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = k(let)$, $\int_C \vec{F}.d\vec{r} = 0 + 1 - \frac{2}{3} = \frac{1}{3}$

SURFACE INTEGRAL:

Surface integral: An integral which is evaluated over a surface is called a surface integral. $\exists$ Surface $S: \ z=\varphi(x,y)$. If $z=\varphi(x,y)$ has a projection on the *xy*-plane, then $\exists$ Vector $\vec{F}(x,y,z)=F_1(x,y,z)\hat{i}+F_2(x,y,z)\hat{j}+F_3(x,y,z)\hat{k}$

The component of $\vec{F}$ along the normal = $\vec{F}.\hat{n}$ = , where $\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{grad\phi}{|grad\phi|}$ is unit normal vector to an element dS . Where $dS=\frac{dx\,dy}{|\hat{n}.k|}$ (for XY- plane),



$dS=\frac{dy\,dz}{|\hat{n}.i|}$ (for YZ- plane), $dS=\frac{dzdx}{|\hat{n}.j|}$ (for ZX- plane).

Surface integral of $\vec{F}$ over S = $\sum \vec{F}.\hat{n}=\iint_S(\vec{F}.\hat{n})ds$

Flux across a surface: The normal surface integral of continuous vector point function $\vec{F}$ on a surface S , i.e. $\iint_S(\vec{F}.\hat{n})d\vec{S}$ is called flux of $\vec{F}$ across the surface **S.** $\vec{F}$ represents the velocity of a liquid.

- If $\iint_S(\vec{F}.\hat{n})ds=0$ then $\vec{F}$ is said to be solenoidal vector point function.

Surface integrals have applications in physics, particularly with the theories of classical electromagnetism.

Q.24. Evaluate $\iint_s(\vec{F}.\hat{n})ds$ where $\vec{F}=18z\,\hat{i}-12x\,\hat{j}+3y\,\hat{k}$ and S is the surface of the plane $2x+3y+6z=12$ in the first octant. **[Jan.06]**

*[**Hint:*** $\hat{n}=\frac{grad\ \phi}{|grad\ \phi|}=\frac{grad(2x+3y+6z-12)}{|2x+3y+6z=12|}=\frac{2i+3j+6k}{7}$, $ds=\frac{dxdy}{|k.\hat{n}|}=\frac{dxdy}{|6/7|}$,

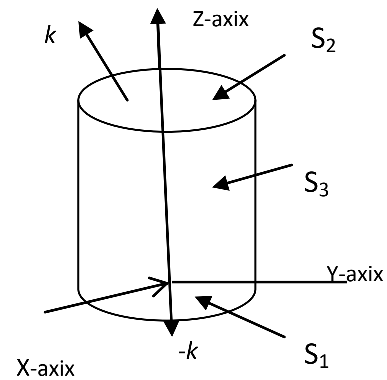
$\vec{F}.\hat{n}=\frac{12}{7}(3-x)$, **Ans**: $\iint_s(\vec{F}.\hat{n})ds=\int_0^6\int_0^{\frac{12-2x}{3}}\frac{12}{7}(3-x)\frac{dxdy}{|6/7|}=24$, for limits put z=0 in *xy* plane

Q.25. Evaluate $\iint_s\vec{F}.\hat{n}\,ds$ where $\vec{F}=z\,\hat{i}+x\,\hat{j}-3y^2z\,\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ *included in first octant by z=0 and z=5.* **[Dec.15]**

Q.26. Evaluate $\iint_s\vec{F}.\hat{n}\,ds$ where $\vec{F}=(x+y^2)\hat{i}-2x\,\hat{j}+2yz\,\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant. **[Dec.16, May. 2019]**

Q.27. Evaluate $\iint_s\vec{F}.\hat{n}\,ds$ where $\vec{F}=4xz\,\hat{i}-y^2\,\hat{j}+yz\,\hat{k}$ and *S* is the surface bounded by the planes *x=0, x=a, y=0, y=a, z=0, z=a*

Q.28. Evaluate $\iint_s \vec{F}\,\hat{n}\,ds$ where $\vec{F} = xi - yj + (z^2 - 1)k$ and S is the surface bounded by the region $x^2 + y^2 = 4$, z=0, and z=1. **[Ans: 4π]**



Hint: Base S₁: Circle $x^2+y^2=4, z=0$, **then** $\vec{F} = xi - yj + (0^2 - 1)k$, $\hat{n} = -k$

, $\iint_{s_1} (\vec{F}.\hat{n})ds_1 = \iint_{s_1} 1 ds_1 = s_1 = \pi(1)^2 = \pi$

On S₂: Circle $x^2+y^2=4, z=1$, **then** $\vec{F} = xi - yj + (1^2 - 1)k$, $\hat{n} = k$,

$\iint_{s_2} (\vec{F}.\hat{n})ds_1 = \iint_{s_2} 0 ds_2 = 0$

On curved surface S₃: $\phi = x^2 + y^2$,$\hat{n} = \dfrac{grad\,\phi}{|grad\,\phi|} = \dfrac{xi + yj}{\sqrt{x^2 + y^2}} = \dfrac{xi + yj}{2}$

$\iint_{s_3} (\vec{F}.\hat{n})ds_3 = \iint_s \dfrac{x^2 - y^2}{2} ds_3 = 0$,

For integration : put x=2cosθ , y=2sinθ, z=z, ds=2dθdz, θ:0 to 2π, z:0 to 1

Q.29. A vector field is given by $\vec{F} = \sin y i + x(1 + \cos y) j$, evaluate the line integral over the circular path given by $x^2 + y^2 = a^2$, $z = 0$. **[Dec.15]**

VOLUME INTEGRAL

A **volume integral** refers to an integral over a 3-dimensional domain, that is, it is a special case of multiple integrals. Volume integrals are especially important in physics for many applications, for example, to calculate flux densities. t can also mean a triple integral within a region $D \subset R^3$ of a function $f(x, y, z)$ is usually written as: $\iiint_D f(x,y,z)dv = \iiint_D f(x,y,z)dxdydz$

Q.30. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_v \nabla.\vec{F} dv$ where V is bounded by x= y= z= 0 and 2x+2y+z=4.

Q.31. If $\vec{A} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$ then evaluate $\iiint_v \vec{A} dv$ where V is the region bounded by the surface x=0 y=0,x=2,y=6, z=x², z=4.

STOKE'S THEOREM: **Let *S* be a regular surface with coherently oriented boundary *C*,**

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{\ell}$$

Q.32. Verify Stoke's theorem for the vector field $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated along the square whose sides are *x=0,y=0,x=a* and *y=a* in the plane *z*=0 .

Q.33. Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by the line $x = \pm a$,y=0 y=b. **[June15,16]**

Q.34. Verify the Stoke's theorem when $\vec{F}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$ and S be the upper half surface of the sphere $x^2+y^2+z^2=1$

Q.35. Verify the Stoke's theorem for the vector field $\vec{F}=z\hat{i}+x\hat{j}+y\hat{k}$ taken over the upper half of the sphere $x^2+y^2+z^2=a^2$ lying above the xy –plane. **[Dec.15]**

Q.36. Verify the Stoke's theorem for the vector field $\vec{F}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$ over the upper half of the surface $x^2+y^2+z^2=1$ bounded by its projection on the xy –plane.

Q.37. Apply Stoke theorem to evaluate $\int_c [(x+y)dx+(2x-z)dy+(z+y)dz]$ where C is the boundary of the triangle with vertices (2,0,0),(0 ,3,0) and (0,0,6). **[Dec.16]**

Q.38. Apply Stoke's theorem to find the value of $\int_c (ydx+zdy+xdz)$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$

Q.39. Verify Stoke's theorem for the vector field $\vec{F}=(x^2-y^2)\hat{i}+2xy\hat{j}$ integrated round the rectangle in the plane z=0 and bounded by the lines x=0,y=0,x=a,y=b.

Q.40. GAUSS DIVERGENCE THEOREM: Let *S* be a regular, positive-oriented closed surface, enclosing a region *V*, $\oiint_S \vec{F}\cdot d\vec{A}=\iiint_V (\nabla\cdot\vec{F})dxdydz=\iiint_v div\vec{F}\,dv$ **.**

Q.41. Verify divergence theorem for $\vec{F}=x^2\hat{i}+z\hat{j}+yz\hat{k}$ taken over the cube bounded by $x=0,x=1,y=0,y=1,z=0,z=1,$

Q.42. Verify divergence theorem for $\vec{F}=x^2\hat{i}+z\hat{j}+yz\hat{k}$ taken over the rectangular paralleopiped by by $x=0,x=a,y=0,y=b,z=0,z=c,$

Q.43. Apply divergence theorem to show that $\iint_s [(x^3-yz)\hat{i}-2x^2y\hat{j}+2\hat{k}]\hat{n}\,ds=\frac{a^5}{3}$,where S denotes the surface of the Cube bounded by the planes $x=0$,$x=a$,$y=0$, $y=a$, $z=0$, $z=a$

Q.44. Evaluate $\iint_s \vec{F}\,\hat{n}\,ds$ where $\vec{F}=x\hat{i}-y\hat{j}+(z^2-1)\hat{k}$ and S is a closed surface bounded by planes $z=0,z=1$and the cylinder $x^2+y^2=4$ (Also verify gauss's divergence theorem.)

Q.45. Verify the Divergence theorem for the function $\vec{F}=2x^2y\hat{i}-y^2\hat{j}+4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$.

Q.46. Using Gauss's Divergence theorem find $\iint_s \vec{F}\,\hat{n}\,ds$ where $\vec{F}=(2x+3z)\hat{i}-(xz+y)\hat{j}+(y^2+2z)\hat{k}$ and S is the surface of the sphere with centre (3,-1,2) and radius 3. **[Nov. 2019]**

GREEN'S THEOREM : Let C be a regular, closed, positively-oriented curve enclosing a region D, $\vec{F}(x,y) = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$.

$$\oint_C F_1(x,y)dx + F_2(x,y)dy = \iint_D \left[\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right] dxdy$$

Q.47. Using Green's theorem to evaluate $\int_c (x^2 y dx + y^3 dy)$ where C is the closed path formed by $y=x$ and $y=x^3$ from(0,0) to (1,1)

Q.48. Verify Green's theorem for $\int_c [(xy + y^2)dx + x^2 dy]$,where C is bounded by y=x and y=x^2.

Q.49. Using Green's theorem to evaluate $\int_c [(xy + x^2)dx + (x^2 + y^2)dy]$ where C is the square formed by the lines $y = \pm 1, x = \pm 1$